INEL 5309- 111 – DSP I – EXAM ONE
Prof. Domingo Rodríguez – Wednesday, March 2, 2005

SOLUTION

Problem One:

(20 points) Obtain the DTFT of the signal $s[n] = x^*[-n] + \cos(2\pi\mu_0 n)e^{+j\omega_d n}$ in terms of the DFT of the signal $x[n]$ which is $X(\omega) = \sum_{n=-\infty}^{n=+\infty} x[n]e^{-j\omega n}$.

Answer to Problem One:

First, we use the linearity property of the DTDT to obtain

$$DTFT\left\{x^*[-n] + \cos(2\pi\mu_0 n)e^{+j\omega_d n}\right\} = DTFT\left\{x^*[-n]\right\} + DTFT\left\{\cos(2\pi\mu_0 n)e^{+j\omega_d n}\right\}$$

We then proceed to compute the DTFT of the first component of the right hand side of the expression above:

$$DTFT\left\{x^*[-n]\right\} = \sum_{n=-\infty}^{n=+\infty} x^*[-n]e^{-j\omega n} = \left(\left(\sum_{n=-\infty}^{n=+\infty} x^*[-n]e^{-j\omega n}\right)^*\right)^*$$

$$\left(\left(\sum_{n=-\infty}^{n=+\infty} x^*[-n]e^{-j\omega n}\right)^*\right)^* = \left(\sum_{n=-\infty}^{n=+\infty}\left(x^*[-n]e^{-j\omega n}\right)^*\right)^* = \left(\sum_{n=-\infty}^{n=+\infty} x[-n]e^{+j\omega n}\right)^*$$

$$\left(\sum_{n=-\infty}^{n=+\infty} x[-n]e^{+j\omega n}\right)^* = \left(\sum_{m=-\infty}^{m=+\infty} x[m]e^{-j\omega m}\right)^* = X^*(\omega)$$

We now proceed to compute the $DTFT\left\{\cos(2\pi\mu_0 n)e^{+j\omega_d n}\right\}$.

We start by using the following identity describing a property of the DTFT:

$$DTFT\left\{e^{+j\lambda n}\right\} = \sum_{k=-\infty}^{k=+\infty} 2\pi\delta(\omega - \lambda + 2\pi k)$$

To prove this identity, we proceed as follows:

$$IDTFT\left(\sum_{k=-\infty}^{k=+\infty} 2\pi\delta(\omega - \lambda + 2\pi k)\right) = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left(\sum_{k=-\infty}^{k=+\infty} 2\pi\delta(\omega - \lambda + 2\pi k)\right)e^{+j\omega n}d\omega$$

We continue the process by interchanging the summation and integral signs

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\left(\sum_{k=-\infty}^{k=+\infty}\delta(\omega - \lambda + 2\pi k)\right)e^{+j\omega n} = \frac{1}{2\pi}\sum_{k=-\infty}^{k=+\infty}\left(\int_{-\pi}^{+\pi} 2\pi\delta(\omega - \lambda + 2\pi k)e^{+j\omega n}d\omega\right)$$

$$\frac{1}{2\pi}\sum_{k=-\infty}^{k=+\infty}\left(\int_{-\pi}^{+\pi} 2\pi\delta(\omega - \lambda + 2\pi k)e^{+j\omega n}d\omega\right) = \int_{-\pi}^{+\pi}\delta(\omega - \lambda)e^{+j\omega n}d\omega = e^{+j\lambda n} \quad \textbf{Q.E.D.}$$

We rewrite the expression $\cos(2\pi\mu_0 n)e^{+j\omega_d n}$ as follows

$$\cos(2\pi\mu_0 n)e^{+j\omega_d n} = \left(\tfrac{1}{2}e^{+j\omega_0 n} + \tfrac{1}{2}e^{-j\omega_0 n}\right)e^{+j\omega_d n} = \tfrac{1}{2}e^{+j(\omega_d+\omega_0)n} + \tfrac{1}{2}e^{+j(\omega_d-\omega_0)n};\quad \omega_0 = 2\pi\mu_0$$

We now let $\sigma = (\omega_d + \omega_0)$ and $\beta = (\omega_d - \omega_0)$ to obtain the following

$$\tfrac{1}{2}e^{+j(\omega_d+\omega_0)n} + \tfrac{1}{2}e^{+j(\omega_d-\omega_0)n} = \tfrac{1}{2}e^{+j\sigma n} + \tfrac{1}{2}e^{+j\beta n}$$

INEL 5309- 111 – DSP I – EXAM ONE
Prof. Domingo Rodríguez – Wednesday, March 2, 2005

SOLUTION

Taking the DTFT of $\frac{1}{2}e^{+j\sigma n}+\frac{1}{2}e^{+j\beta n}$ results in the following

$$DTFT\left\{\tfrac{1}{2}e^{+j\sigma n}+\tfrac{1}{2}e^{+j\beta n}\right\}=DTFT\left\{\tfrac{1}{2}e^{+j\sigma n}\right\}+DTFT\left\{\tfrac{1}{2}e^{+j\beta n}\right\}$$

$$DTFT\left\{\tfrac{1}{2}e^{+j\sigma n}\right\}+DTFT\left\{\tfrac{1}{2}e^{+j\beta n}\right\}=\sum_{k=-\infty}^{k=+\infty}\pi\delta\left(\omega-\sigma+2\pi k\right)+\sum_{k=-\infty}^{k=+\infty}\pi\delta\left(\omega-\beta+2\pi k\right)$$

We, thus, get our final expression for the desired Fourier transform

$$DTFT\left\{s[n]\right\}=S\left(\omega\right)=DTFT\left\{x^{*}[-n]+\cos(2\pi\mu_0 n)e^{+j\omega_d n}\right\}=X^{*}\left(\omega\right)+DTFT\left\{\cos(2\pi\mu_0 n)e^{+j\omega_d n}\right\}$$

$$S\left(\omega\right)=X^{*}\left(\omega\right)+DTFT\left\{\cos(2\pi\mu_0 n)e^{+j\omega_d n}\right\}=X^{*}\left(\omega\right)+\sum_{k=-\infty}^{k=+\infty}\pi\delta\left(\omega-\sigma+2\pi k\right)+\sum_{k=-\infty}^{k=+\infty}\pi\delta\left(\omega-\beta+2\pi k\right)$$

Problem Two:

2a.- (20 points) Compute the DFT of the signal $x[n]=\left[\cos(\frac{2\pi n}{4})\right]^2, n\in Z_4$

Answer to Problem 2a:

We first express $x[n]=\left[\cos(\frac{2\pi n}{4})\right]^2, n\in Z_4$ as $x[n]=g[n]g[n], n\in Z_4$ where $g[n]$ is just a discrete cosine signal; that is, $g[n]=\cos(\frac{2\pi n}{4}), n\in Z_4$:

$$g=\begin{bmatrix}\cos(\frac{2\pi(0)}{4})=+1\\ \cos(\frac{2\pi(1)}{4})=0\\ \cos(\frac{2\pi(2)}{4})=-1\\ \cos(\frac{2\pi(3)}{4})=0\end{bmatrix},\qquad x=g^2=\begin{bmatrix}+1\\0\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}+1\\0\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\1\\0\end{bmatrix},$$

Thus, we have $x[n]=\left\{x[0],x[1],x[2],x[3]\right\}=\left\{1,0,1,0\right\}$ as the input signal. We continue to compute the DFT as follows

$$X[k]=\sum_{n=0}^{n=3}x[n]e^{-j\frac{2\pi}{4}kn}=x[o]+x[2]e^{-j\pi k}=1+(1)e^{-j\pi k};\quad k\in Z_4$$

Evaluating for values of $k\in Z_4$, we obtain

$$X[k]=\left\{X[0],X[1],X[2],X[3]\right\}=\left\{2,0,2,0\right\}$$

2b.- (20 points) Compute the cyclic convolution of $x[n]$ and $h[n]$ if $x[n]=h[n]=\left[\cos(\frac{2\pi n}{4})\right]^2, n\in Z_4$ and provide a matrix representation of the cyclic convolution operation.

Answer to Problem 2b:

We write the equation for the cyclic convolution operation of $x[n]$ and $h[n]$.

$$y[n]=\sum_{k=0}^{3}x[k]h\left[\left\langle n-k\right\rangle_4\right];\quad k\in Z_4$$

INEL 5309- 111 – DSP I – EXAM ONE
Prof. Domingo Rodríguez – Wednesday, March 2, 2005

SOLUTION

We proceed to write the matrix representation of this operation

$$H_4 \cdot x = y = \begin{bmatrix} h[0] & h[3] & h[2] & h[1] \\ h[1] & h[0] & h[3] & h[2] \\ h[2] & h[1] & h[0] & h[3] \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Problem Three:

3a.- (15 points) Determine if the following linear system is a filter:

$$T\{x[n]\} = x[-n]$$

Answer to Problem 3a:

Since the system is linear, we must determine if it is time invariant. If we have the system equation $T\{x[n]\} = y[n]$, then, a system is time-invariant if

$$T\{x[n-n_0]\} = y[n-n_0]$$

We proceed by computing the right hand side of the above equation

$$y[n-n_0] = x[-(n-n_0)] = x[-n+n_0]$$

We compute the left hand side by first introducing an intermediate step. Let $T\{x[n]\} = x[-n]$. We then have

$$T\{g[n]\} = g[-n] = x[-n-n_0]$$

We conclude that the system is NOT A FILTER.

Answer to Problem 3b:

3b.- (10 points) Determine if the following system is a filter:

$$T\{x[n]\} = e^{x(t+1)+x(t-1)}$$

We can test for homogeneity, a necessary condition for linearity:

$$T\{ax[n]\} = aT\{x[n]\}$$

We first compute the right hand side of the identity

$$aT\{x[n]\} = ae^{x(t+1)+x(t-1)}$$

Let $T\{ax[n]\} = g[n]$. We then have

$$T\{g[n]\} = e^{g(t+1)+g(t-1)} = e^{ax(t+1)+ax(t-1)}$$

We conclude that the system is NOT A FILTER.

INEL 5309- 111 – DSP I – EXAM ONE
Prof. Domingo Rodríguez – Wednesday, March 2, 2005

SOLUTION

Problem Four:

(15 points) Classify the following signals (digital, real, complex, continuous, etc.) Name all the properties that the signal may possess.

Answer to Problem Four:

a).- $x:\mathbb{R}\to\mathbb{Z}_4;\quad x(t)=(t+1)/(t+1);\quad t\in\mathbb{R}$

a).- The signal is CONTINUOUS and DIGITAL

b).- $y\in L\left(\mathbb{Z}_8\right);\quad y[n]=e^{+j\frac{2\pi}{8}n};\quad n\in\mathbb{Z}_{128}$

b).- The signal is DISCRETE and COMPLEX

c).- $s\in L\left(\mathbb{Z}\right);\quad \mathrm{s}[n]=\left\langle n\right\rangle_N;\quad n\in\mathbb{Z}$

c).- The signal is DISCRETE and DIGITAL

d).- $g:\mathbb{Z}_7\to\mathbb{Z}_7;\quad g[n]=\left\langle n^2\right\rangle_7;\quad n\in\mathbb{Z}_7$

d).-The signal is DISCRETE and DIGITAL